Math 4200
Friday August 28
1.2-1.3 Algebra and geometry of complex arithmetic from Wednesday's notes; introduction to complex plane transformations section 1.3, in today's notes. We'll pick up in Wednesday's notes where we left off, and continue into today's.

Announcements:

Warm-up exercise:

Section 1.3, "basic" complex functions.

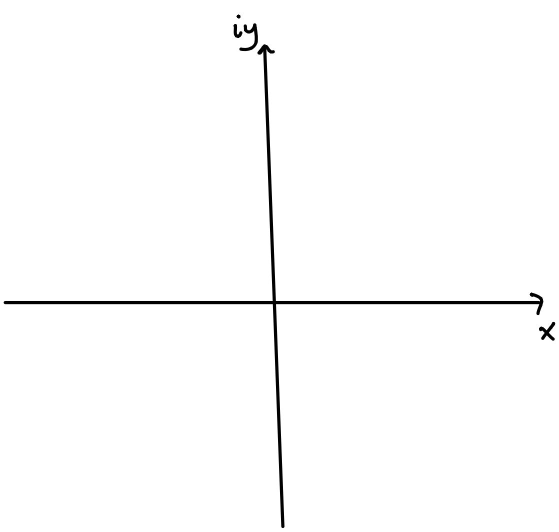
<u>Example 1</u> $f : \mathbb{C} \to \mathbb{C},\ f(z) = a\,z + b \quad (a, b \in \mathbb{C})$.

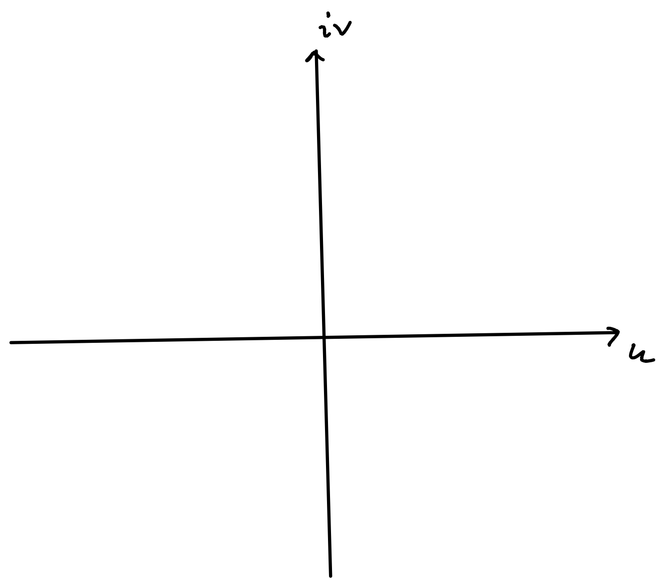
Using polar form makes this transformation easy to understand geometrically. Write

$$z = |\,z\,|\, \mathrm{e}^{i\ arg\ z},$$
$$a = |\,a\,|\, \mathrm{e}^{i\ arg\ a} \qquad .$$

and compute $f(z)$ in order to interpret it as the composition of a rotation, a scaling, and then a translation.

Illustrate the general situation above for the transformation $f(z) = (1 + i)\ z - 2$.





When we start discussing complex differentiability in section 1.5 we will want to go back and forth between discussions for complex-valued functions $f : \mathbb{C} \rightarrow \mathbb{C}$ and the mathematically equivalent discussions for related real functions $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, so that we can take advantage of what you know about continuity and differentiability for real vector-valued functions, Math 3220.

The correspondence is that for each

$$f : \mathbb{C} \rightarrow \mathbb{C}$$
$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$$

with $u(x, y) = \mathrm{Re}(f(z))$, $v(x, y) = \mathrm{Im}(f(z)$, there is

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$F(x, y) = (u(x, y), v(x, y)).$$

And for each $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ there is a corresponding $f : \mathbb{C} \rightarrow \mathbb{C}$.

If we apply this correspondence to the example on the previous page, with $f(z) = a\,z + b$ and

$$z = x + i\,y$$
$$a = a_1 + i\,a_2$$
$$b = b_1 + i\,b_2,$$

then

$$f(x + i\,y) = \left(a_1 + i\,a_2\right)(x + i\,y) + \left(b_1 + i\,b_2\right)$$
$$= \left(a_1\,x - a_2\,y + b_1\right) + i\left(a_2\,x + a_1 y + b_2\right)$$

If we write the corresponding real and imaginary components of F in column form we recognize certain affine transformations from linear algebra:

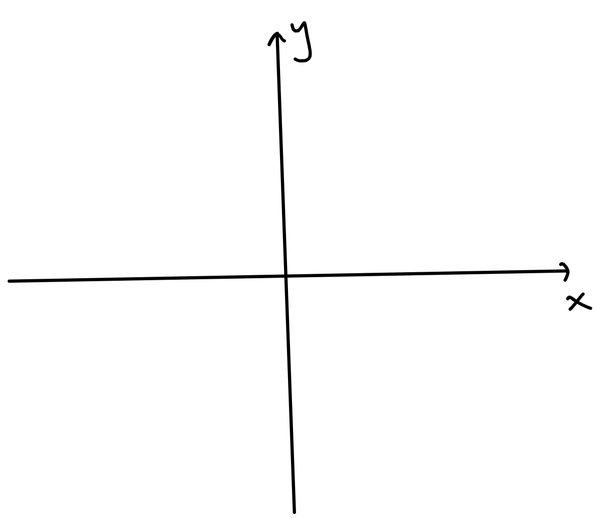
$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} a_1\,x - a_2\,y + b_1 \\ a_2\,x + a_1 y + b_2 \end{bmatrix} = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

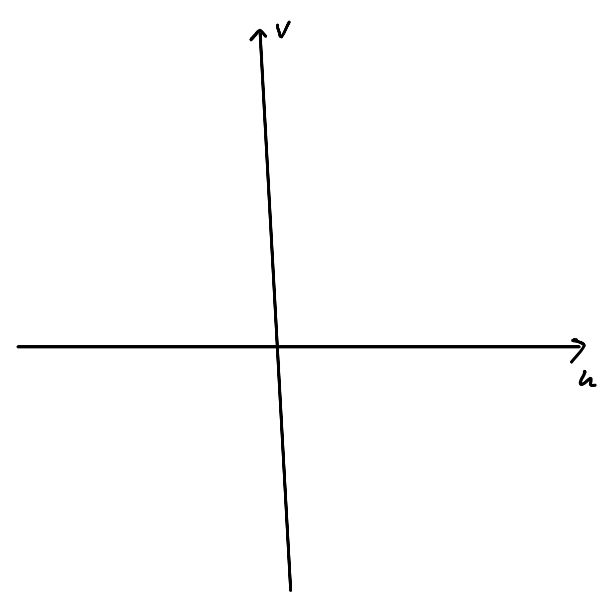
Recall your geometric matrix transformations of the plane from your linear algebra class to describe F geometrically on the next page, which will be equivalent to how we described f on the previous page.

$$f(z) = a\,z + b$$
$$f(z) = |\,a\,|\,e^{i \arg a}\,|\,z\,|\,e^{i \arg z} + b$$

$$\begin{aligned} f(x + i\,y) &= (a_1 + i\,a_2)(x + i\,y) + (b_1 + i\,b_2) \\ &= (a_1\,x - a_2\,y + b_1) + i(a_2\,x + a_1 y + b_2) \end{aligned}$$

$$F(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} a_1\,x - a_2\,y + b_1 \\ a_2\,x + a_1 y + b_2 \end{bmatrix} = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

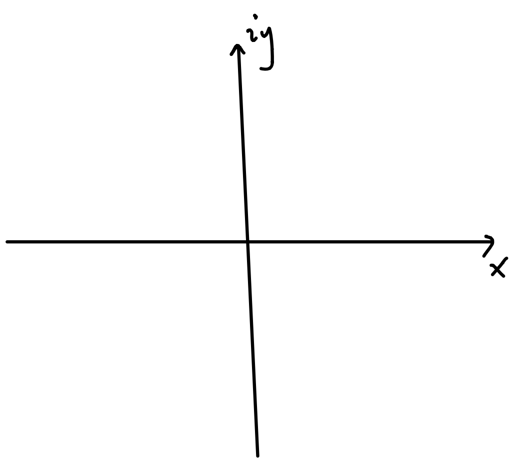


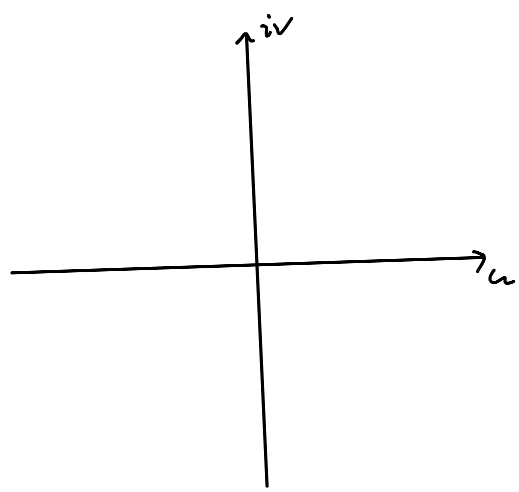


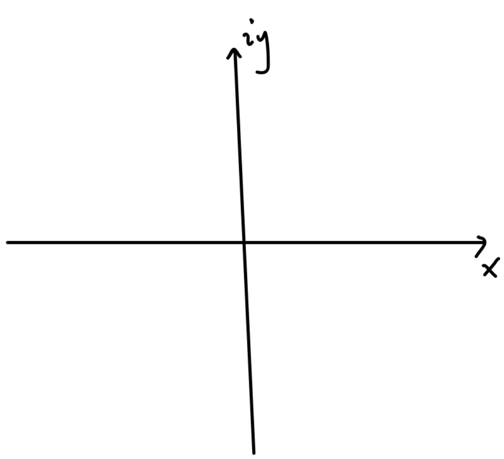
Example 2

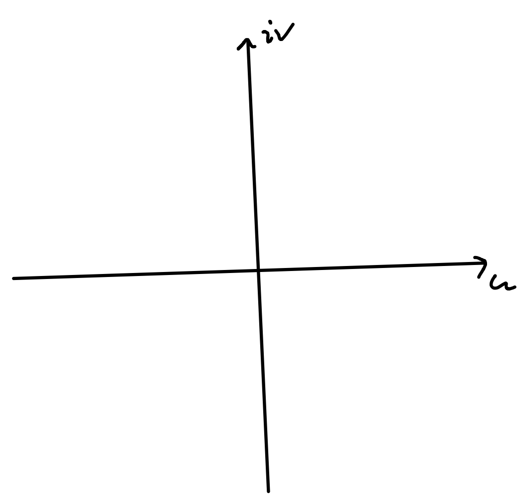
$$f(z) = z^2$$
$$f\left(r\,e^{i\,\theta}\right) = r^2\,e^{i\,2\,\theta}$$

Discuss and sketch how f transforms $z-$plane into a (mostly) twice-covered $w-$plane.
For $w = z^2$ discuss possible continuous inverse functions $z = \sqrt{w}$, and corresponding domains.





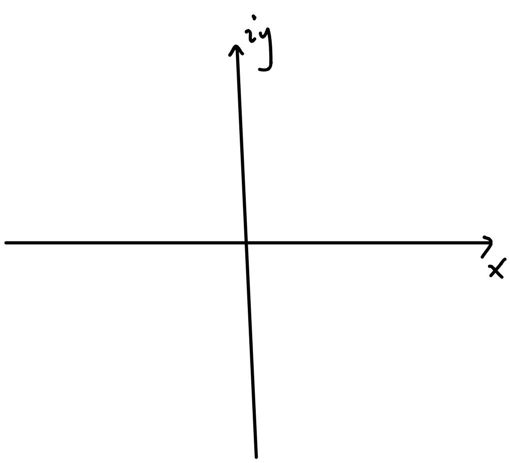


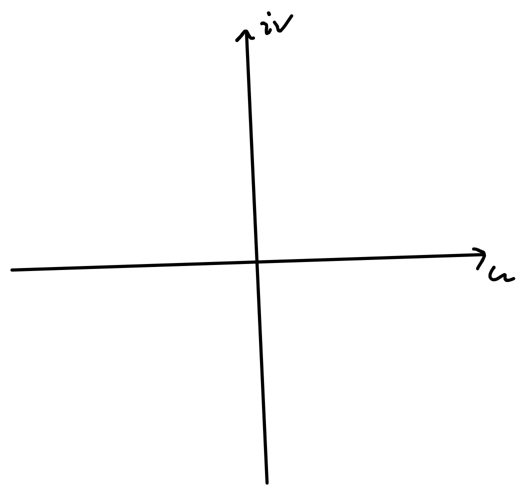


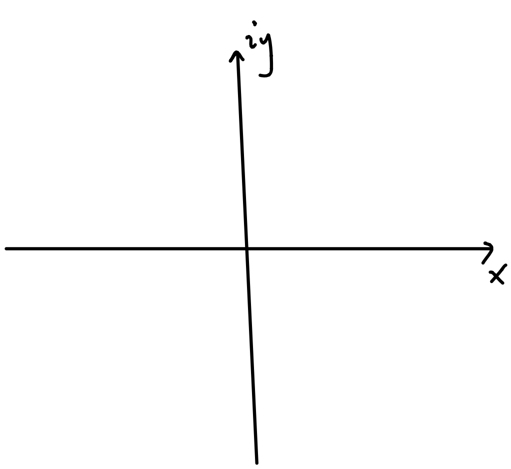
Example 3

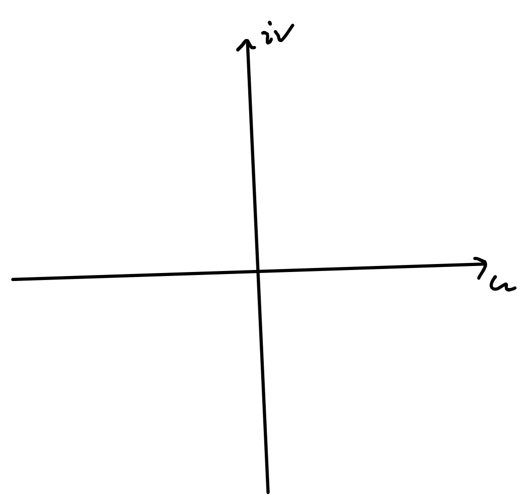
$$f(z) = z^3$$
$$f\left(r\,e^{i\theta}\right) = r^3\,e^{i\theta}.$$

Discuss and sketch how f transforms the $z-$plane into a (mostly) triple-covered $w-$plane. For $w = z^3$ discuss possible continuous inverse functions $z = \sqrt[3]{w}$, and corresponding domains.









Example 4 For $z = x + i\,y, \quad x, y \in \mathbb{R}$

$$f(z) = e^{z} = e^{x + i\,y} := e^{x}\, e^{i\,y}$$

Discuss and sketch how f transforms the $z-$plane into an infinitely covered $w-$plane.
For $w = e^{z}$ discuss possible continuous inverse functions $z = \log(w)$, and corresponding domains.